UNIVERSAL WISER PUBLISHER

Asymptotic Solutions of Fifth Order Overdamped-Oscillatory Nonlinear Systems

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Abstract: Combo overdamp-oscillatory system plays an important role in natural phenomena in many engineering problems. In this paper, fifth order nonlinear damped-oscillatory differential system is studied to investigate an asymptotic analytical approximate solution in the fashion of overdamp-oscillations via an extension of the Krylov-Bogoliubov-Mitropolskii (KBM) method. The proposed method is demonstrated by its applications on a Duffing oscillators in the combined form of overdamp and oscillatory effects. The result obtained by the presented extended technique good agreement with the numerical solutions of the fourth order Runge-Kutta method.
Keywords: cubic non-linearity, overdamped-oscillation, perturbation method, eigen-values

1. Introduction

Nonlinear oscillating phenomena have a lots of significant applications in several aspects of physics as well as other natural and various fields of applied sciences specially in engineering fields. The last few decades, various efficient and powerful methods have been established by a diverse groups of dynamic researchers to construct the analytical approximate solutions of physically important non-linear equations [1-24]. Among the methods, the Krylov-Bogoliubov-Mitropolskii (KBM) (Bogoliubov and Mitropolskii [1]; Krylov and Bogoliubov [2]) method is a vastly reliable technique to investigate an analytical approximate solutions of nonlinear systems with both small and large nonlinearity. However, the process was invented for resulting of the periodic solutions of second order nonlinear systems with small nonlinearities, Popov [3] prolonged the method to explore the solutions of damped oscillatory nonlinear model. Murty and Deekshatulu [4] presented a process via Bogoliubov's scheme to procure the transient response of over-damped nonlinear model. Owing to physical impact, Mendelson [5] re-experienced Popov's results. Murty [6] presented a unified KBM technique for finding approximate solutions of second order nonlinear models, which covers the un-damped, damped, and over-damped cases. Bojadziev and Hung [7] affianced the KBM method to explore approximate solutions of damped oscillations demonstrated by a 3-dimensional time dependent structure. Sattar [8] reputed an asymptotic solution of a second order critically damped nonlinear model. Alam [9] projected a novel perturbation scheme to find the analytical approximate solution of nonlinear model with large damping and then Alam [10] prolonged the method for n-th order nonlinear model which covers over-damped and critically damped. Alam and Sattar [11] presented a unified method for finding solution of third order damped oscillatory and over-damped nonlinear model. Akbar et al. [12] investigated a procedure for deciphering fourth order over-damped nonlinear model. Later, Akbar et al. [13] prolonged the procedure for damped oscillatory nonlinear model in the case when the four eigen-values are complex conjugates. Akbar [14] also investigated solution of fourth order nonlinear systems in which two of the eigen-values are real and negative and the rest of the two are complex conjugates. Recently, Akbar and Siddique [15, 16] investigated schemes to derive the analytical approximate solutions (damp-oscillatory, overdamp model respectively) of fifth-order weakly nonlinear oscillatory model by extending the KBM method. More other researches are performed in diverse scheme on nonlinear ODEs / PDEs [17-24].

In this research, we aimed to investigate asymptotic approximate solutions of fifth order overdamped oscillatory nonlinear model involving two of the eigen-values are complex conjugates and the other three are real and negative.

DOI: https://doi.org/10.37256/cm.142020496

2. The method

In this section, we consider a fifth order weakly nonlinear overdamped-oscillatory ordinary differential system in the form of

$$\frac{d^5x}{dt^5}+\sum_{i=1}^{4}c_i\frac{d^ix}{dt^i}+c_5x=-\varepsilon f(x,\ t), \tag{1}$$

where ε is a tiny valued parameter, $f(x, t)$ is the nonlinear function, c_i ; i = 1, 2,..., 5 are the characteristic parameters of the system defined by $c_1=\sum_{i=1}^{5}\lambda_i,\ c_2=\sum_{\substack{i,j=1\\ i\neq j}}^{5}\lambda_i\lambda_j,\ c_3=\sum_{\substack{i,j,k=1\\ i\neq j\neq k}}^{5}\lambda_i\lambda_j\lambda_k,\ c_4=\sum_{\substack{i,j,k,l=1\\ i\neq j\neq k\neq l}}^{5}\lambda_i\lambda_j\lambda_k\lambda_l$ and $c_5=\prod_{i=1}^{5}\lambda_i$ where $-\lambda_1$, $-\lambda_2$, $-\lambda_3$, $-\lambda_4$, $-\lambda_5$, are the eigenvalues of the unperturbed equation (1).

When $\varepsilon = 0$ i.e, unperturbed equation yields the solution,

$$x(t,\ 0)=\sum_{j=1}^{5}a_{j,0}e^{-\lambda_j t}, \tag{2}$$

where $a_{j,0}$, j = 1, 2,..., 5 are random constants.

For the constraint $\varepsilon \neq 0$, we look for a solution in accordance with Shamsul [18], of the form

$$x(t,\ \varepsilon)=\sum_{j=1}^{5}a_j(t)e^{-\lambda_j t}+\varepsilon u_1(a_1,\ a_2,\cdots,\ a_5,\ t)+\cdots, \tag{3}$$

where each a_j ; j = 1, 2,..., 5, frequently agreed with the conditions

$$\dot{a}_j(t)=\varepsilon A_j(a_1,\ a_2,\cdots,\ a_5,\ t)+\varepsilon^2\cdots. \tag{4}$$

Keeping our attention on the first little terms 1, 2,..., m in the series expansions (3) and (4), we estimate the functions u_1 and A_j ; j = 1, 2,..., 5 such that a_j ; j = 1, 2,..., 5, execution in (3) and (4), fulfill the differential equation (1) with an accuracy of ε^{m+1}. Nevertheless, the solution can be acquired up to the correctness of any order of approximation, by reason of the rapidly-arising complication expressions for the manipulation of the formulae, the target solution, in general, impeding to first order. To manipulate these unknown parameters, it is expected that u_1 exclude fundamental terms which are involved in the series (3) at order ε^0.

Differentiating $x(t, \varepsilon)$ five times with respect to t and plugging $x(t, \varepsilon)$ and their derivatives in (1) together with the relations (4) and, then equating the coefficients of ε, we attain

$$\prod_{j=1}^{5}(\frac{d}{dt}+\lambda_j)u_1+\sum_{j=1}^{5}e^{-\lambda_j t}(\prod_{k=1,j\neq k}^{5}(\frac{d}{dt}-\lambda_j+\lambda_k))A_j=-f^{(0)}(a_1,\ a_2,\cdots,\ a_5,\ t), \tag{5}$$

where $f^{(0)}=f(x_0)$ and $x_0=\sum_{j=1}^{5}a_j(t)e^{-\lambda_j t}$.

The function $f^{(0)}$ can be prolonged in a Taylor series (see Murty and Deekshatulu [4] for details) as:

$$f^{(0)}=\sum_{m_1=-\infty\cdots m_5=-\infty}^{\infty,\ldots,\infty}F_{m_1,\cdots m_5}\sum_{i=1}^{5}a_i^{m_i}e^{-(m_1\lambda_1+m_2\lambda_2+\cdots+m_5\lambda_5)t}.$$

To obtain the solution of eq.(1), it has been proposed in (Shamsul 2003) [13] that u_1 exclude the fundamental terms. To do this we have to separated the eq.(5) into six equations for unknown functions u_1 and A_j ; j = 1, 2,..., 5 (see [17, 18] for details).

Substituting the functional values and equating the coefficients of $e^{-\lambda_j t}$; $j = 1, 2, ..., 5$, we obtain

$$e^{-\lambda_1 t}\sum_{i=2}^{5}(\frac{d}{dt}-\lambda_1+\lambda_i)A_1=-\sum_{\substack{m_1=-\infty,\cdots,m_5=-\infty\\ m_3=m_4,\,m_1=m_2+1}}^{\infty.....\infty}F_{m_1,\cdots m_5}\sum_{i=1}^{5}a_i^{m_i}e^{-(m_1\lambda_1+m_2\lambda_2+\cdots+m_5\lambda_5)t}. \tag{6}$$

$$e^{-\lambda_2 t}\sum_{i=1,i\neq 2}^{5}(\frac{d}{dt}-\lambda_2+\lambda_i)A_2=-\sum_{\substack{m_1=-\infty,\cdots,m_5=-\infty\\ m_3=m_4,\,m_1=m_2-1}}^{\infty.....\infty}F_{m_1,\cdots m_5}\sum_{i=1}^{5}a_i^{m_i}e^{-(m_1\lambda_1+m_2\lambda_2+\cdots+m_5\lambda_5)t}. \tag{7}$$

$$e^{-\lambda_3 t}\sum_{i=1,i\neq 3}^{5}(\frac{d}{dt}-\lambda_3+\lambda_i)A_3=-\sum_{\substack{m_1=-\infty,\cdots,m_5=-\infty\\ m_1=m_2,\,m_3=m_4+1}}^{\infty.....\infty}F_{m_1,\cdots m_5}\sum_{i=1}^{5}a_i^{m_i}e^{-(m_1\lambda_1+m_2\lambda_2+\cdots+m_5\lambda_5)t}. \tag{8}$$

$$e^{-\lambda_4 t}\sum_{i=1,i\neq 4}^{5}(\frac{d}{dt}-\lambda_4+\lambda_i)A_4=-\sum_{\substack{m_1=-\infty,m_5=-\infty\\ m_1=m_2,\,m_3=m_4-1}}^{\infty.....\infty}F_{m_1,\cdots m_5}\sum_{i=1}^{5}a_i^{m_i}e^{-(m_1\lambda_1+m_2\lambda_2+.........+m_5\lambda_5)t}. \tag{9}$$

$$e^{-\lambda_5 t}\sum_{i=1}^{4}(\frac{d}{dt}-\lambda_5+\lambda_i)A_5=-\sum_{\substack{m_1=-\infty\cdots m_5=-\infty\\ m_1=m_2,\,m_3=m_4}}^{\infty.....\infty}F_{m_1,\cdots m_5}\sum_{i=1}^{5}a_i^{m_i}e^{-(m_1\lambda_1+m_2\lambda_2+.........+m_5\lambda_5)t}. \tag{10}$$

and

$$\sum_{i=1}^{5}(\frac{d}{dt}+\lambda_i)u_1=-\sum_{m_1=-\infty\cdots m_5=-\infty.}^{\infty.....\infty}F_{m_1,\cdots m_5}\sum_{i=1}^{5}a_i^{m_i}e^{-(m_1\lambda_1+m_2\lambda_2+.........+m_5\lambda_5)t}. \tag{11}$$

where u' avoid the terms for $m_1 = m_2 \pm 1$, $m_3 = m_4 \pm 1$, $m_1 = m_2$, $m_3 = m_4$.

Solving (6) to (11), we obtain $A_1, A_2, ..., A_5$ and u.

We shall be able to transform (3) to the exact formal KBM [1, 2] solution by relieving $a_1 = \frac{a}{2}e^{\varphi_1}$, $a_2 = \frac{a}{2}e^{-\varphi_1}$, $a_3 = \frac{b}{2}e^{i\varphi_2}$ and $a_4 = \frac{b}{2}e^{-i\varphi_2}$. Herein a, b are amplitudes and φ_1, φ_2 are phase variables.

3. Example

As an example of the above procedure, we are going to consider the Duffing type equation of fifth order

$$\frac{d^5x}{dt^5}+\sum_{i=1}^{4}c_i\frac{d^ix}{dt^i}+c_5x=-\varepsilon x^3, \tag{12}$$

here $f(x, t) = x^3$.

We have $f^{(0)} = (\sum_{i=1}^{5} a_i e^{-\lambda_i t})^3$,

or

$$f^{(0)} = a_1^3 e^{-3\lambda_1 t} + 3a_1^2 a_2 e^{-(2\lambda_1+\lambda_2)t} + 3a_1 a_2^2 e^{-(\lambda_1+2\lambda_2)t} + a_2^3 e^{-3\lambda_2 t}$$

$$+ 3a_1^2 a_3 e^{-(2\lambda_1+\lambda_3)t} + 3a_1^2 a_4 e^{-(2\lambda_1+\lambda_4)t} + 3a_1^2 a_5 e^{-(2\lambda_1+\lambda_5)t}$$

$$+ 6a_1 a_2 a_3 e^{-(\lambda_1+\lambda_2+\lambda_3)t} + 6a_1 a_2 a_4 e^{-(\lambda_1+\lambda_2+\lambda_4)t} + 6a_1 a_2 a_5 e^{-(\lambda_1+\lambda_2+\lambda_5)t}$$

$$+ 3a_2^2 a_3 e^{-(2\lambda_2+\lambda_3)t} + 3a_2^2 a_4 e^{-(2\lambda_2+\lambda_4)t} + 3a_2^2 a_5 e^{-(2\lambda_2+\lambda_5)t}$$

$$+3a_1 a_3^2 e^{-(\lambda_1+2\lambda_3)t} + 3a_1 a_4^2 e^{-(\lambda_1+2\lambda_4)t} + 3a_1 a_5^2 e^{-(\lambda_1+2\lambda_5)t}$$

$$+ 6a_1 a_3 a_4 e^{-(\lambda_1+\lambda_3+\lambda_4)t} + 6a_1 a_3 a_5 e^{-(\lambda_1+\lambda_3+\lambda_5)t} + 6a_1 a_4 a_5 e^{-(\lambda_1+\lambda_4+\lambda_5)t} \quad . \qquad (13)$$

$$+3a_2 a_3^2 e^{-(\lambda_2+2\lambda_3)t} + 3a_2 a_4^2 e^{-(\lambda_2+2\lambda_4)t} + 3a_2 a_5^2 e^{-(\lambda_2+2\lambda_5)t}$$

$$+ 6a_2 a_3 a_4 e^{-(\lambda_2+\lambda_3+\lambda_4)t} + 6a_2 a_4 a_5 e^{-(\lambda_2+\lambda_4+\lambda_5)t} + 6a_2 a_3 a_5 e^{-(\lambda_2+\lambda_3+\lambda_5)t}$$

$$+ a_3^3 e^{-3\lambda_3 t} + 3a_3^2 a_4 e^{-(2\lambda_3+\lambda_4)t} + 3a_3 a_4^2 e^{-(\lambda_3+2\lambda_4)t} + a_4^3 e^{-3\lambda_4 t}$$

$$+ 3a_3^2 a_5 e^{-(2\lambda_3+\lambda_5)t} + 6a_3 a_4 a_5 e^{-(\lambda_3+\lambda_4+\lambda_5)t} + 3a_4^2 a_5 e^{-(2\lambda_4+\lambda_5)t}$$

$$+3a_3 a_5^2 e^{-(\lambda_3+2\lambda_5)t} + 3a_4 a_5^2 e^{-(\lambda_4+2\lambda_5)t} + 3a_5^3 e^{-3\lambda_5 t}$$

Thus the equations (6) to (11) takes the form

$$e^{-\lambda_1 t} \sum_{i=2}^{5} (\frac{d}{dt} - \lambda_1 + \lambda_i) A_1 = -3a_1^2 a_2 e^{-(2\lambda_1+\lambda_2)t} - 6a_1 a_3 a_4 e^{-(\lambda_1+\lambda_3+\lambda_4)t}. \qquad (14)$$

$$e^{-\lambda_2 t} \sum_{i=1, i\neq 2}^{5} (\frac{d}{dt} - \lambda_2 + \lambda_i) A_2 = -3a_1 a_2^2 e^{-(\lambda_1+2\lambda_2)t} - 6a_2 a_3 a_4 e^{-(\lambda_2+\lambda_3+\lambda_4)t}. \qquad (15)$$

$$e^{-\lambda_3 t} \sum_{i=1, i\neq 3}^{5} (\frac{d}{dt} - \lambda_3 + \lambda_i) A_3 = -3a_3^2 a_4 e^{-(2\lambda_3+\lambda_4)t} - 6a_1 a_2 a_3 e^{-(\lambda_1+\lambda_2+\lambda_3)t}. \qquad (16)$$

$$e^{-\lambda_4 t} \sum_{i=1, i\neq 4}^{5} (\frac{d}{dt} - \lambda_4 + \lambda_i) A_4 = -3a_3 a_4^2 e^{-(\lambda_3+2\lambda_4)t} - 6a_1 a_2 a_4 e^{-(\lambda_1+\lambda_2+\lambda_4)t}. \qquad (17)$$

$$e^{-\lambda_5 t} \sum_{i=1}^{4} (\frac{d}{dt} - \lambda_5 + \lambda_i) A_5 = -6a_1 a_2 a_5 e^{-(\lambda_1+\lambda_2+\lambda_5)t} - 6a_3 a_4 a_5 e^{-(\lambda_3+\lambda_4+\lambda_5)t}. \qquad (18)$$

and

$$\sum_{i=1}^{5}(\frac{d}{dt}+\lambda_i)u_1 = -(a_1^3e^{-3\lambda_1 t} + a_2^3e^{-3\lambda_2 t} + 3a_1^2a_3e^{-(2\lambda_1+\lambda_3)t} + 3a_1^2a_4e^{-(2\lambda_1+\lambda_4)t} + 3a_1^2a_5e^{-(2\lambda_1+\lambda_5)t}$$

$$+3a_2^2a_3e^{-(2\lambda_2+\lambda_3)t} + 3a_2^2a_4e^{-(2\lambda_2+\lambda_4)t} + 3a_2^2a_5e^{-(2\lambda_2+\lambda_5)t}$$

$$+3a_1a_3^2e^{-(\lambda_1+2\lambda_3)t} + 3a_1a_4^2e^{-(\lambda_1+2\lambda_4)t} + 3a_1a_5^2e^{-(\lambda_1+2\lambda_5)t}$$

$$+6a_1a_3a_5e^{-(\lambda_1+\lambda_3+\lambda_5)t} + 6a_1a_4a_5e^{-(\lambda_1+\lambda_4+\lambda_5)t} \tag{19}$$

$$+3a_2a_3^2e^{-(\lambda_2+2\lambda_3)t} + 3a_2a_4^2e^{-(\lambda_2+2\lambda_4)t} + 3a_2a_5^2e^{-(\lambda_2+2\lambda_5)t}$$

$$+6a_2a_4a_5e^{-(\lambda_2+\lambda_4+\lambda_5)t} + 6a_2a_3a_5e^{-(\lambda_2+\lambda_3+\lambda_5)t}$$

$$+a_3^3e^{-3\lambda_3 t} + a_4^3e^{-3\lambda_4 t} + 3a_3^2a_5e^{-(2\lambda_3+\lambda_5)t} + 3a_4^2a_5e^{-(2\lambda_4+\lambda_5)t}$$

$$+3a_3a_5^2e^{-(\lambda_3+2\lambda_5)t} + 3a_4a_5^2e^{-(\lambda_4+2\lambda_5)t} + 3a_5^3e^{-3\lambda_5 t})$$

.

Again solving the equations (14) to (18) and inserting $\lambda_1 = k_1 - \omega_1$, $\lambda_2 = k_1 + \omega_1$, $\lambda_3 = k_2 - i\omega_2$, $\lambda_4 = k_2 + i\omega_2$ and $\lambda_5 = \xi$, we obtain

$$A_1 = -\frac{3a_1^2a_2e^{-2k_1t}}{2(k_1-\omega_1)\{(3k_1-k_2-\omega_1)^2+\omega_2^2)\}\{(3k_1-\xi)-\omega_1\}}$$

$$-\frac{6a_1a_3a_4e^{-2k_2t}}{2(k_2-\omega_1)\{(k_1+k_2-\omega_1)^2+\omega_2^2)\}\{(k_1+2k_2-\xi)-\omega_1\}},$$

$$A_2 = -\frac{3a_1a_2^2e^{-2k_1t}}{2(k_1+\omega_1)\{(3k_1-k_2+\omega_1)^2+\omega_2^2)\}\{(3k_1-\xi)+\omega_1\}}$$

$$-\frac{6a_2a_3a_4e^{-2k_2t}}{2(k_2+\omega_1)\{(k_1+k_2+\omega_1)^2+\omega_2^2)\}\{(k_1+2k_2-\xi)+\omega_1\}},$$

$$A_3 = -\frac{3a_3^2a_4e^{-2k_2t}}{2(k_2-i\omega_2)\{(3k_2-k_1)+(\omega_1-i\omega_2)\}\{(3k_2-k_1)-(\omega_1+i\omega_2)\}\{(3k_2-\xi)-i\omega_2\}}$$

$$-\frac{6a_1a_2a_3e^{-2k_1t}}{2(k_1-i\omega_2)\{(k_1+k_2)+(\omega_1-i\omega_2)\}\{(k_1+k_2)-(\omega_1+i\omega_2)\}\{(2k_1+k_2-\xi)-i\omega_2\}},$$

$$A_4 = -\frac{3a_3a_4^2e^{-2k_2t}}{2(k_2+i\omega_2)\{(3k_2-k_1)+(\omega_1+i\omega_2)\}\{(3k_2-k_1)-(\omega_1-i\omega_2)\}\{(3k_2-\xi)+i\omega_2\}},$$

$$-\frac{6a_1a_2a_4e^{-2k_1t}}{2(k_1+i\omega_2)\{(k_1+k_2)+(\omega_1+i\omega_2)\}\{(k_1+k_2)-(\omega_1-i\omega_2)\}\{(2k_1+k_2-\xi)+i\omega_2\}}$$

$$A_5 = -\frac{6a_1a_2a_5e^{-2k_1t}}{\{(k_1+\xi)^2-\omega_1^2)\}\{(2k_1-k_2+\xi)^2+\omega_2^2\}} - \frac{6a_3a_4a_5e^{-2k_2t}}{\{(k_2+\xi)^2+\omega_2^2)\}\{(2k_2-k_1+\xi)^2+\omega_2^2\}}.$$

Here u_1 is a small perturbed term and has also very minor contribution in the solution but it is laborious to solve (19) for u_1. So we neglect the scheming of u_1. Now inserting A_j ; $j = 1, 2,..., 5$ in the equations (4) and using

$a_1 = \frac{1}{2}ae^{\phi_1}$, $a_2 = \frac{1}{2}ae^{-\phi_1}$, $a_3 = \frac{1}{2}be^{i\phi_2}$, $a_4 = \frac{1}{2}be^{-i\phi_2}$ and $a_5 = c$ we obtain

$$\dot{a} = \varepsilon(l_1a^3e^{-2k_1t} + l_2ab^2e^{-2k_2t}),\ \dot{b} = \varepsilon(m_1b^3e^{-2k_2t} + m_2a^2be^{-2k_1t}),$$

$$\dot{\phi}_1 = \varepsilon(n_1a^2e^{-2k_1t} + n_2b^2e^{-2k_2t}),\ \dot{\phi}_2 = \varepsilon(q_1b^2e^{-2k_2t} + q_2a^2e^{-2k_1t}),$$

and

$$\dot{c} = \varepsilon(p_1a^2ce^{-2k_1t} + p_2b^2ce^{-2k_2t}), \tag{20}$$

where

$$l_1 = -\frac{3}{16}\left\{\frac{\begin{gathered}(k_1+\omega_1)(3k_1-\xi+\omega_1)((3k_1-k_2)^2+\omega_1^2+\omega_2^2+2\omega_1(3k_1-k_2))\\ +(k_1-k_2)(3k_1-\xi-\omega_1)((3k_1-k_2)^2+\omega_1^2+\omega_2^2-2\omega_1(3k_1-k_2))\end{gathered}}{(k_1^2-\omega_1^2)\{(3k_1-\xi)^2-\omega_1^2\}\{(3k_1-k_2)^4+(\omega_1^2+\omega_2^2)^2-(\omega_1^2+\omega_2^2)(3k_1-k_2)^2\}}\right\},$$

$$l_2 = -\frac{3}{4}\left\{\frac{\begin{gathered}(k_1-\omega_1)(k_1+2k_2-\xi-\omega_1)\{(k_1+k_2)^2+\omega_1^2+\omega_2^2-2\omega_1(k_1+k_2)\}\\ +(k_1+\omega_1)(k_1+2k_2-\xi-\omega_1)\{(k_1+k_2)^2+\omega_1^2+\omega_2^2+2\omega_1(k_1+k_2)\}\end{gathered}}{(k_2^2-\omega_1^2)\{(k_1+2k_1-\xi)^2-\omega_1^2\}\{(k_1+k_2)^4+(\omega_1^2+\omega_2^2)^2-2(\omega_1^2+\omega_2^2)(k_1^2+k_2^2)\}}\right\},$$

$$n_1 = -\frac{3}{16}\left\{\frac{\begin{gathered}(k_1+\omega_1)(3k_{12}-\xi+\omega_1)\{(3k_1-k_2)^2+\omega_1^2+\omega_2^2+2\omega_1(3k_1-k_2)\}\\ -(k_1-k_2)(3k_1-\xi-\omega_1)\{(3k_1-k_2)^2+\omega_1^2+\omega_2^2-2\omega_1(3k_1-k_2)\end{gathered}}{(k_1^2-\omega_1^2)\{(3k_1-\xi)^2-\omega_1^2\}\{(3k_2-k_1)^4+(\omega_1^2+\omega_2^2)^2-2(\omega_1^2+\omega_2^2)(3k_1-k_2)^2\}}\right\},$$

$$n_2 = -\frac{3}{8}\left\{\frac{\begin{matrix}(k_2+\omega_1)(k_1+2k_2-\xi+\omega_1)\{(k_1+k_2)^2+\omega_1^2+\omega_2^2+2\omega_1(k_1+k_2)\}\\ -(k_2-\omega_1)(k_1+2k_2-\xi-\omega_1)\{(k_1+k_2)^2+\omega_1^2+\omega_2^2-2\omega_1(k_1+k_2)\end{matrix}}{(k_2^2-\omega_1^2)\{k_1+2k_2-\xi)^2-\omega_1^2\}\{(k_2+k_1)^4+(\omega_1^2+\omega_2^2)^2-2(\omega_1^2+\omega_2^2)(k_1+k_2)^2\}}\right\},$$

$$m_1 = -\frac{3}{8}\left\{\frac{\begin{matrix}(3k_1^2-k_1k_2+\omega_1^2+\omega_1\omega_2)(6k_1\omega_1-k_2\omega_1-3k_1\omega_2-\omega_1\xi+\omega_2\xi)+\\ (4k_1\omega_1+k_1\omega_2-k_2\omega_1)(9k_1^2-3k_1k_2-3k_1\xi+k_2\xi+\omega_1^2-\omega_1\omega_2)\end{matrix}}{(k_1^2-\omega_1^2)\{(3k_1-k_2)^2-(\omega_1+\omega_2)^2\}\{(3k_1-k_2)^2-(\omega_1-\omega_2)^2\}\{(3k_1-\xi)^2-\omega_1^2\}}\right\},$$

$$m_2 = -\frac{3}{4}\left\{\frac{\begin{matrix}(k_1k_2+k_2^2+\omega_1^2-\omega_1\omega_2)(2k_1\omega_1+3k_2\omega_1+k_1\omega_2+2k_2\omega_2-\omega_1\xi-\omega_2\xi)-\\ (2k_2\omega_1-k_2\omega_2+k_1\omega_1)\{(k_1+k_2)(k_1+2k_2-\xi)+\omega_1^2+\omega_1\omega_2\}\end{matrix}}{(k_2^2-\omega_1^2)\{(k_1+k_2)^2-(\omega_1-\omega_2)^2\}\{(k_1+k_2)^2-(\omega_1+\omega_2)^2\}\{(k_1+2k_2-\xi)^2-\omega_1^2\}}\right\},$$

$$q_1 = -\frac{3}{8}\left\{\frac{\begin{matrix}(3k_2^2-k_1k_2+\omega_2^2+\omega_1\omega_2)(6k_2\omega_2-k_1\omega_2-3k_2\omega_1+\omega_1\xi-\omega_2\xi)-\\ (4k_2\omega_2-k_1\omega_2+k_2\omega_1)(9k_2^2-3k_1k_2-3k_2\xi+k_1\xi+\omega_2^2-\omega_1\omega_2)\end{matrix}}{(k_2^2-\omega_2^2)\{(3k_2-k_1)^2-(\omega_1-\omega_2)^2\}\{(3k_2-k_1)^2-(\omega_1+\omega_2)^2\}\{(3k_2-\xi)^2-\omega_2^2\}}\right\},$$

$$q_2 = -\frac{3}{4}\left\{\frac{\begin{matrix}(k_1^2+k_1k_2+\omega_2^2+\omega_1\omega_2)(2k_2\omega_2+3k_1\omega_2-2k_1\omega_1-k_2\omega_1+\omega_1\xi-\omega_2\xi)\\ -(k_1\omega_1+2k_1\omega_2+k_2\omega_2)\{(k_1+k_2)(2k_1+k_2-\xi)+\omega_2^2-\omega_1\omega_2\}\end{matrix}}{(k_1^2-\omega_2^2)\{(k_2+k_1)^2-(\omega_1-\omega_2)^2\}\{(k_2+k_1)^2-(\omega_1+\omega_2)^2\}\{(2k_1+k_2-\xi)^2-\omega_2^2\}}\right\},$$

$$p_1 = -\frac{3}{2}\frac{1}{\{(k_1+\xi)^2-\omega_1^2)\}\{(2k_1-k_2+\xi)^2+\omega_2^2\}},$$

and

$$p_2 = -\frac{3}{2}\frac{1}{\{(k_2+\xi)^2+\omega_2^2)\}\{(2k_2-k_1+\xi)^2+\omega_2^2\}}.$$

Equations (20) are nonlinear and have no exact solutions. We can solve (20) taking a, b, c, φ_1 and φ_2 are constants in the right-hand sides as ε is very minor, $\dot{a}$, $\dot{b}$, $\dot{c}$, $\dot{\varphi}_1$ and $\dot{\varphi}_2$ are petite varying function of time. This conjecture was charity by Murty et al. [4, 17] to solve the similar nonlinear equations. The solution is thus

$$a(t) = a_0 + \varepsilon(l_1 a_0{}^3 \frac{(1 - e^{-2k_1 t})}{2k_1} + l_2 a_0 b_0{}^2 \frac{(1 - e^{-2k_2 t})}{2k_2}),$$

$$b(t) = b_0 + \varepsilon(m_1 b_0{}^3 \frac{(1 - e^{-2k_2 t})}{2k_2} + m_2 a_0{}^2 b_0 \frac{(1 - e^{-2k_1 t})}{2k_1}),$$

$$\varphi_1(t) = \varphi_{1,0} + \varepsilon(n_1 a_0{}^2 \frac{(1 - e^{-2k_1 t})}{2k_1} + n_2 b_0{}^2 \frac{(1 - e^{-2k_2 t})}{2k_2}),$$

$$\varphi_2(t) = \varphi_{2,0} + \varepsilon(q_1 b_0{}^2 \frac{(1 - e^{-2k_2 t})}{2k_2} + q_2 a_0{}^2 \frac{(1 - e^{-2k_1 t})}{2k_1}),$$

and

$$c(t) = c_0 + \varepsilon(p_1 a_0{}^2 c_0 \frac{(1 - e^{-2k_1 t})}{2k_1} + p_2 b_0{}^2 c_0 \frac{(1 - e^{-2k_2 t})}{2k_2}). \tag{21}$$

Finally, we obtain the solution in the form

$$x(t) = a\cosh(\omega_1 t + \varphi_1) + b\cos(\omega_2 t + \varphi_2) + ce^{-\xi t}. \tag{22}$$

Here (22) is the first order approximate solution of eq. (12), where a, b, c, φ_1 and φ_2 are given by (21).

4. Result and discussion

In order to check the accuracy of an analytical approximate solution obtained based on KBM method, we have compared our obtained results (by perturbation) to those obtained by the fourth order Runge-Kutta method for different sets of initial conditions. Beside this, we have also computed the Pearson correlation between the perturbation results and the corresponding numerical results and shows that they are strongly correlated. From the figures we observed that our perturbation solution agrees with numerical results suitably for different initial conditions.

At first, for $k_1 = 1/3$, $k_2 = 0.25$, $\omega_1 = 0.15$, $\omega_2 = \sqrt{5}$, $\xi = 0.5$ and $\varepsilon = 0.1$, $x(t, \varepsilon)$ has been computed (22), in which a, b, c, φ_1 and φ_2 by the equation (21) with initial conditions

$$a_0 = 0.25,\ b_0 = 0.25,\ c_0 = 0.25,\ \varphi_{1,0} = \frac{\pi}{6} \text{ and } \varphi_{2,0} = \frac{\pi}{6}$$

i.e, $x(0) = 0.751566,\ \frac{dx(0)}{dt} = -0.550235,\ \frac{d^2x(0)}{dt^2} = -0.82637$

$$\frac{d^3x(0)}{dx^3} = 2.101099 \text{ and } \frac{d^4x(0)}{dt^4} = 3.655849.$$

In this section, the perturbation results obtained by the solution (22) and the corresponding numerical results obtained by a fourth order Runge-Kutta method with a small time increment 0.5, are plotted (Figure 1). The correlation between the results is 0.999037.

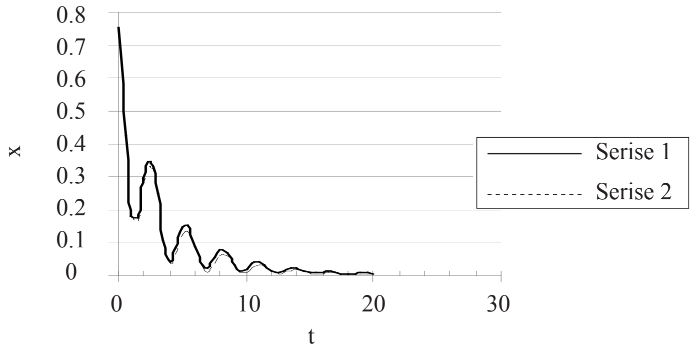

Figure 1. Perturbation solution plotted by solid line and numerical solution plotted by dotted line

Finally, for $k_1 = 1/3$, $k_2 = 0.25$, $\omega_1 = \frac{\pi}{6}$, $\omega_2 = \frac{\pi}{6}$, $\xi = 0.5$ and $\varepsilon = 0.1$, $x(t, \varepsilon)$ has been computed (22), in which a, b, c, ϕ_1 and ϕ_2 by the equation (21) with initial conditions

$$a_0 = 0.5,\ b_0 = 0.5,\ c_0 = 0.5,\ \phi_{1,0} = \frac{\pi}{6} \text{ and } \phi_{2,0} = \frac{\pi}{6}$$

i.e., $x(0) = 1.503132$, $\frac{dx(0)}{dt} = -1.203219$, $\frac{d^2x(0)}{dt^2} = -1.556837$,

$$\frac{d^3x(0)}{dt^3} = 4.132355 \text{ and } \frac{d^4x(0)}{dt^4} = 7.362476.$$

In this section, the perturbation results obtained by the solution (22) and the corresponding numerical results obtained by a fourth order Runge-Kutta method with a small time increment 0.5, are plotted (Figure 2). The correlation between the results is 0.998661.

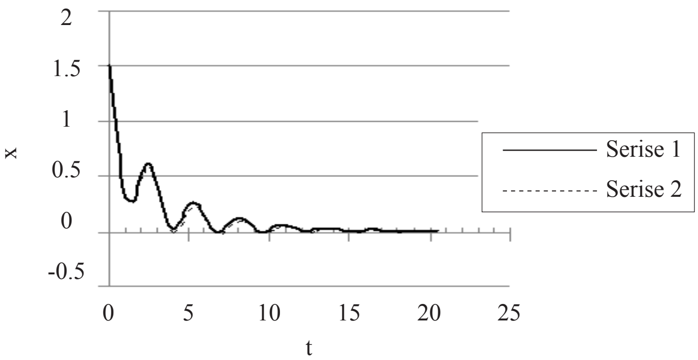

Figure 2. Perturbation solution plotted by solid line and numerical solution plotted dotted line

5. Conclusion

An asymptotic analytical approximate solution of a fifth order overdamped-oscillatory nonlinear differential systems is presented via the theory of KBM [1, 2] method in this article. The correlation coefficients are also calculated between the derived solution of presented method and the fourth order Runge-Kutta method of the consider problem. The accuracy of obtained result is compared with numerical result for different initial conditions and founded excellent coincidence in each conditions. We also observed that the results are strongly correlated.

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